A Power-Log Utility Model for Pricing Stock-Index Options

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Abstract

The Black-Scholes-Merton option pricing model works well for at the money options, but not for in the money and out of the money options. We use investor preferences modeled with a Power-Log utility function, which is a variation of prospect theory, along with investor perceptions about market volatility, to calculate prices for in the money and out of the money stock-index options. The prices from our model are closer to market prices than Black-Scholes-Merton prices.

Keywords: Option Pricing, Index Options, Power-Log Utility, Prospect Theory

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Option pricing models such as Black-Scholes (1973) and Merton (1973), do not incorporate investor preferences, and that can lead to wide discrepancies between the model price and the observed market price for in the money and out of the money options. Our model incorporates investor preferences explicitly with a Power-Log utility function, which is a variation of prospect theory, and produces prices for in the money and out of the money S&P500 index call and put options that are closer to market prices than those obtained from the Black-Scholes-Merton model. Our method is fundamentally different from, and on a sounder theoretical footing, than other models that have been created to improve on B-S-M pricing, since they focus on improving the stochastic process that generates asset returns but ignore investor preferences. In addition, for pricing the options we use a very long history of monthly S&P500 returns, 864 months, which incorporates the effects of jumps and stochastic volatility observed in equity returns.

The market data we use for S&P500 put and call options, which are European options, is from a snapshot of the Schwab website on November 16, 2022, at 2:55 pm Eastern Standard Time in the US, for options that expire in 30 days on December 16, 2022. The bid and ask quotes are live quotes. The S&P500 index value at the time of the snapshot was 3,966.04, its dividend yield was 1.75%, and the one-month US treasury yield was 3.81% (www.treasury.gov). To highlight inconsistencies in the B-S-M model, we calculated B-S-M option implied volatilities for selected call options, with the bid-ask midpoint as our measure of the market price of the option. The B-S-M model for the price of a European call option,

$$C = Se^{-DT}N(d_1) - Xe^{-rT}N(d_2) \quad (1)$$

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + (r - D + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where,

C = call option value

S = price of underlying asset

X = exercise price of call option

T = time to call option expiration

r = risk-free rate

D = continuous dividend yield on underlying asset

σ = volatility, or standard deviation of underlying asset return

N(d) = cumulative standard normal probability of d

Fig. 1 shows S&P500 call option implied volatilities on November 16, 2022. It implies that if market prices are the true measures of value, investors would have to use different measures of volatility for the underlying asset, when calculating B-S-M call option prices with different strikes. This inconsistency is a shortcoming of the B-S-M model, since there is only one underlying asset, and its volatility should be the same for all strikes. It is widely accepted that the B-S-M model provides a reasonably accurate measure of value for at the money options, so the B-S-M volatility implied by the market price of an at the money option on the S&P500 index, gives us a good way of divining investor perceptions of market volatility. These perceptions can change quickly, as shown by the precipitous decline in volatility from November 16, 2022, to July 19, 2023, during which time the volatility index, VIX, dropped from 24.39% to 13.64%.

To model investor preferences, we use Power-Log utility functions (Kale [2006]), which are a variation of prospect theory (Kahneman and Tversky [1979], Tversky and Kahneman [1991]) that has been used to explain investor behavior. Kale [2006], Kale and Sheth [2016], and Kale and Lim [2020] show that the Power-Log utility functions are very effective in asset allocation and portfolio selection applications. They produce portfolios that conform closely to investor preferences for maximizing portfolio growth and controlling downside risk, and perform better than portfolio optimization using either mean-variance analysis, or power utility functions.

To price in the money and out of the money call options on the S&P500 index, we treat a call option as an instrument for achieving equity exposure. This equity exposure is not simply leveraged equity, but the returns generated by the call option are dependent on the returns generated by the S&P500 index. To price the option, we find the call price that produces a portfolio consisting of the call option and a treasury, that has the same expected utility (Savage [1964], Von Neumann and Morgenstern [1944]) as an investment in the S&P500 index, which represents equity investment. That still leaves us with an identification problem, since a range of call option prices and combinations of investment weights in the treasury-call portfolio will result in the same expected utility. We solve this identification problem by using another element, that is the coin of the realm in options markets, namely volatility. To price the call option uniquely, we match the volatility of the treasury-call portfolio to the volatility of the S&P500 index investment, in addition to matching their expected utilities.

Power-Log Utility Functions

The following description of Power-Log utility functions is based on Kale [2006], Kale and Sheth [2016], and Kale and Lim [2020]. Power-Log utility functions start by incorporating the central idea of behavioral economics and prospect theory, that investors view gains and losses asymmetrically (Kahneman and Tversky [1979], Tversky and Kahneman's [1991]), and then superimpose risk aversion across the entire range of returns so that Power-Log utility functions conform to the Friedman-Savage [1948] axioms for risk-averse utility functions. Best and Grauer (2016 and 2017) show that prospect theory's S-shaped utility function's risk-seeking behavior on the downside results in optimal portfolios, "…that are extremely unstable over different decision horizons and with annual data are prone to risk bankruptcy." Taking bigger risks in an attempt to recoup losses, as implied by prospect theory, is behavior that can lead to ruin, and is not common among investors. By replacing prospect theory's risk-seeking behavior on the downside with risk-averse behavior, Power-Log utility functions convert prospect theory's descriptive model of speculation, into a normative model for investment that conforms closely to investor preferences.

A Power-Log utility function is defined as,

$$U = \ln(1+r) \quad \text{for } r \geq 0 \tag{2}$$

$$= \frac{1}{\gamma}(1+r)^{\gamma} \quad \text{for } r < 0$$

where,

r asset, or portfolio return

γ downside power, less than or equal to 0

Fig. 2 shows examples of Power-Log utility functions, where the utility for losses is modeled with a power utility function and the utility for gains is modeled with the log utility function. The log utility function, which is functionally equivalent to the Kelly criterion (Kelly [1956]), is well known for its portfolio growth-maximization property. It is equivalent to a power utility function with power 0 (Grauer and Hakansson [1982]). Selecting a downside power of zero for the Power-Log utility function is equivalent to using a log utility function for losses, which makes the entire utility function a log utility function. Power-Log utility functions with more negative downside powers, such as -3 and -15, and so on, build in progressively larger penalties for losses, and thus provide a continuum of utility functions that accommodate the full range of investor preferences for controlling losses, while leaving the utility for gains unchanged in the form of the log utility function.

S&P500 Index Returns

We chose to use an empirical distribution of monthly S&P500 returns instead of assuming a theoretical distribution that tries to approximate the underlying distribution. We start with seventy-two years of monthly price returns and total returns from 1950 through 2021, to ensure that the data includes bull markets, bear markets and flat markets, and includes several business cycles. We use total returns to measure the return to equity, and price returns to estimate expiration values for the call options. The 864 monthly return observations give us a well-defined frequency distribution for the purposes of our option pricing model. The data is from Morningstar's Large Stock series, which tracks the S&P500 index.

Table 1 shows selected S&P500 index returns and summary statistics. Fig. 3 shows the frequency distribution for the 1950-2021 history, highlighting the occasional large losses in equities, which lead to negative skewness in equity returns. The negative skewness of the S&P500 returns deviates noticeably from the positive skewness of the lognormal distribution, that is often assumed for equity asset returns and underlies the B-S-M option pricing model. To forecast S&P500 returns for the coming month, we incorporate the volatility of the S&P500 index as perceived by traders and investors in the options market into the forecast of S&P500 returns. We adjust the historical monthly returns so that the volatility of the return forecast reflects the B-S-M implied volatility of the at the money call option on the S&P500 index, and leave the average return and shape of the distribution unchanged.

The market data for options is from a snapshot of the Schwab website on November 16, 2022, at 2:55 pm Eastern Standard Time in the US, as described above. From the available options, we selected options that expire in 30 days on December 16, 2022. The S&P500 index value at that time of the snapshot was 3,966.04, its dividend yield was 1.75%, and the one-month US treasury yield was 3.81% (www.treasury.gov). For estimating market volatility, we chose the closest to the money call option with an exercise price of 3,965 as our approximation for the at the money call option. It had, a bid of 106.00 and ask of 106.6, giving us a bid-ask midpoint of 106.30, that we use as our estimate of its market price. This market price gave us an annual B-S-M implied volatility of 22.63%, which is less than the VIX of 24.39% at that time.

To create a forecast of S&P500 price returns for the coming month, we adjust the historical S&P500 monthly price returns so that the monthly return distribution has a volatility of 6.53% to correspond to the annual ATM implied volatility of 22.63%. Similarly, we adjust the S&P500 monthly total returns, accounting for the slightly higher volatility of total returns over price returns, to forecast total returns for the coming month with a volatility of 6.54%. The average return and shape of the monthly return distributions remains unchanged.

Market's Downside Sensitivity

To identify the downside power that corresponds to collective investor preferences in the S&P500 options market, we create a portfolio consisting of a one-month treasury and the ATM call option, such that volatility of this portfolio equals that of the S&P500 total returns index forecast. For that we need a forecast of the ATM call option returns. We use the S&P500 index value of 3,966.04, and the forecast of S&P500 monthly price returns, to forecast the S&P500 index values on December 16, the option expiration date. The ATM call option strike of 3,965 and the forecast of S&P500 index values, in turn give us a forecast of the option expiration values shown in Table 2. The 106.3 price of the ATM call option and the forecast of its expiration values, give us the ATM call return forecast shown in the last column of Table 2.

The standard deviation of the monthly ATM call option return is 144.05%. To match the standard deviation of 6.54% of S&P500 monthly total returns, the treasury-call portfolio requires an investment of 95.46% in the one-month treasury, which is riskless, and an investment of 4.54% in the ATM call option. With these investment weights, the distribution of ATM call option returns, and a monthly treasury yield of 0.31%, which corresponds to the annualized one-month treasury rate of 3.81%, we construct the monthly return distribution of the treasury-call portfolio.

Once we have the return distribution for the volatility matched treasury-call portfolio, we use an iterative procedure for finding the downside power in the Power-Log utility function which gives us the same expected utilities for the treasury-call portfolio and the S&P500 index investment. At 2:55 pm Eastern Standard Time, on November 16, 2022, that turned out to be a downside power of -3.5. This downside power represents the collective downside sensitivity of investors in the call options market at that time. As market conditions change, this downside sensitivity will change as well.

Call Option Prices

The downside power of -3.5 gives us a well-defined Power-Log utility function, with which we can calculate expected utilities to price the ATM call for different S&P500 spot prices, and prices for in the money and out of the money call options for different strike prices.

Fig. 4 shows the Power-Log Utility OPM prices for the ATM Call Option. To calculate the call price for a given S&P500 index value, we use the forecast of the S&P500 price return distribution to forecast the S&P500 price distribution and corresponding exercise values for the call. Next, we use an iterative procedure to find the call price such that resulting call return distribution produces a treasury-call portfolio that has the same volatility and expected utility as the S&P500 index investment.

For comparing the performance of Power-Log Utility OPM and B-S-M prices relative to market prices, we selected call options with substantial open interest, and strike prices from 3,500 to 4,500, a range that is over 10% above and below the ATM strike. For each call option we calculated the Power-Log Utility OPM price by using the procedure described in the paragraph above. The call prices are shown in Table 3, along with B-S-M prices calculated with the ATM implied volatility. For comparison, Table 3 also shows B-S-M prices using VIX in place of the ATM implied volatility. The model prices are higher than market prices for out of the money call options, and lower than the market prices for in the money call options. Fig. 5 shows the difference between model and market prices. Table 3 and Fig. 5 show that Power-Log Utility OPM prices are closer to market prices than B-S-M prices calculated using ATM call implied volatility, except for the ATM strike of 3,965, where they are the same by design. Also, except for the strike of 3,845, where the B-S-M price calculated with VIX equals the market price, and for a few strikes near it, the Power-Log Utility OPM prices are closer to market prices than B-S-M with VIX prices. The calculation of VIX appears to have effectively targeted the call option with strike 3,845 for the index calculation.

Spot checking data for call options with 30 days to expiration in 2023, gave us results that are similar to those for the December 16, 2022, expiration, with Power-Log Utility OPM call prices that are closer to market prices than B-S-M prices. The Power-Log Utility OPM displays consistently superior performance, in spite of the steep drop in the VIX from its November 16, 2022, value of 24.39% to its July 19, 2023 value of 13.64%.

Put Option Prices

We use put-call parity for European options to calculate the value of put options on the S&P500 index,

$$P = C + Xe^{-rT} - Se^{-DT} \tag{3}$$

where the call price, C, is from the Power-Log Utility OPM. Since the ATM put implied volatility of 22.24% at 2:55 pm Eastern Standard Time on November 16, 2022, was not identical to the ATM call implied volatility of 22.63%, we use the ATM put implied volatility for forecasting S&P500 returns, calculate Power-Log Utility OPM call option values as described in the sections above, and then use the call option values to calculate corresponding put values using put-call parity. Fig. 6 shows the Power-Log Utility OPM put prices for the ATM option with strike price 3,965.

To compare the performance of the Power-Log Utility OPM with the B-S-M model, we calculate put prices for selected strikes from 3,500 to 4,500. The results are similar to those for call options. Fig. 7 shows the difference between model prices and market prices. It shows that Power-Log Utility OPM prices are closer to market prices than B-S-M prices calculated using ATM put implied volatility, except for the ATM strike of 3,965, where they are the same by design. Also, except for the strike of 3,815, where the B-S-M price calculated with VIX equals the market price, and for a few strikes near it, the Power-Log Utility OPM prices are closer to market prices than B-S-M with VIX prices. The VIX calculation appears to have effectively targeted the put option with strike 3,815.

Spot checking data for put options with 30 days to expiration in 2023, gave us results that are similar to those for the December 16, 2022 expiration; Power-Log Utility OPM prices are closer to market prices than B-S-M prices. The Power-Log Utility OPM displays consistently superior performance for both call and put index options.

Conclusion

Option pricing models such as Black-Scholes (1973) and Merton (1973), do not incorporate any information about investor preferences, which can lead to wide discrepancies between the model prices and observed market prices. The Power-Log Utility option pricing model incorporates investor preferences with a Power-Log utility function (Kale [2006]), which is a variation of prospect theory (Kahneman and Tversky [1979], Tversky and Kahneman [1991]), that converts prospect theory's model of speculative behavior into a normative model for asset allocation, portfolio selection, and now option pricing. To price a call option on the S&P500 index, we treat it as an instrument for achieving equity exposure and find the call price that produces a portfolio consisting of the call option and a treasury, that has the same volatility and expected utility as an investment in the S&P500 index. To price a put option on the S&P500 index, we use put-call parity for European options and the corresponding call price from the Power-Log Utility OPM for a given strike. Our method is fundamentally different from, and on a sounder theoretical footing, than other models that have been created to improve on B-S-M pricing, since they focus on improving the stochastic process that generates asset returns but ignore investor preferences. In addition, for pricing the options we use a very long history of monthly S&P500 returns, 864 months, which incorporates the effects of jumps and stochastic volatility observed in equity returns.

For S&P500 index options with 30 days to expiration, the Power-Log Utility OPM prices are closer to market prices than Black-Scholes-Merton prices calculated with the ATM implied volatility, except for the ATM strike, where they are equal by design. They are also closer to market prices than B-S-M prices calculated with VIX, except in the neighborhood of the strike that is effectively targeted for the VIX calculation.

Like B-S-M, the Power-Log Utility OPM relies on the underlying asset's volatility as an input. To calculate a B-S-M option price, the investor has to provide an estimate of that volatility for the life of the option, while the Power-Log Utility OPM, as described above, uses the ATM option implied volatility to price in the money and out of the money options. If an investor has reason to believe that their own estimate of future underlying asset volatility is better than the ATM option implied volatility, then that estimate can be used in the Power-Log Utility OPM for pricing at the money, in the money and out of the money options.

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Table 1

S&P500 Index Monthly Returns, 1950-2021 (%)

Observation Number	Month	Price Return	Total Return
1	1950-01	1.73	1.97
2	1950-02	1.00	1.99
3	1950-03	0.41	0.70
4	1950-04	4.51	4.86
5	1950-05	3.93	5.09
⋮	⋮	⋮	⋮
860	2021-08	2.90	3.04
861	2021-09	-4.76	-4.65
862	2021-10	6.91	7.01
863	2021-11	-0.83	-0.69
864	2021-12	4.36	4.48
Minimum		-21.76	-21.54
Maximum		16.30	16.81
Average		0.74	1.01
Std. Deviation		4.15	4.16
Skewness		-0.43	-0.43
Kurtosis		1.72	1.75

Table 2

At The Money S&P500 Call Return Forecast

Observation Number	S&P500 Return Forecast (%)	S&P500 Price Forecast	ATM Call Expiration Value ($)	ATM Call Return Forecast (%)
1	2.30	4,057.07	92.07	-13.39
2	1.14	4,011.35	46.35	-56.40
3	0.21	3,974.53	9.53	-91.04
4	6.67	4,230.46	265.46	149.73
5	5.75	4,194.16	229.16	115.58
⋮	⋮	⋮	⋮	⋮
860	4.13	4,129.93	164.93	55.15
861	-7.90	3,652.59	0.00	-100.00
862	10.44	4,380.29	415.29	290.68
863	-1.74	3,897.23	0.00	-100.00
864	6.43	4,221.11	256.11	140.93

* S&P500 spot price: 3,966.04

* ATM call strike price: 3,965

Table 3

S&P500 Call Option Prices ($)

2:55 pm Eastern Standard Time, November 16, 2022

Strike	Market Price	P-L Utility OPM Price	B-S-M Price with ATM-IV	B-S-M Price with VIX
3,500	482.60	477.97	473.60	474.88
3,600	388.95	385.46	378.46	381.07
3,700	299.70	297.44	289.04	293.44
3,800	217.65	215.87	209.02	215.28
3,900	145.95	145.16	141.91	149.51
3,950	114.85	114.62	113.94	121.85
3,965	106.30	106.30	106.30	114.26
4,000	87.60	88.25	89.83	97.78
4,100	45.70	48.53	52.76	60.01
4,200	20.55	23.46	28.66	34.49
4,250	13.10	16.07	20.51	25.50
4,300	8.20	11.04	14.39	18.54
4,400	3.25	5.09	6.67	9.33
4,480	1.70	2.64	3.41	5.14
4,500	1.48	2.28	2.86	4.40

* ATM call strike price: 3,965

Fig. 1

Black-Scholes-Merton Call Implied Volatility (%)
2:55 pm Eastern Standard Time, November 16, 2022

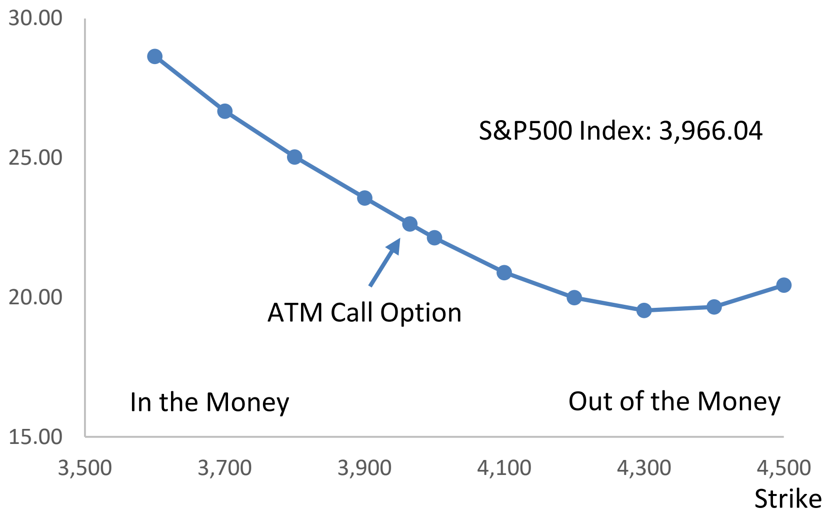

Fig. 2

Power-Log Utility Function with Downside Powers 0, -3 and -15

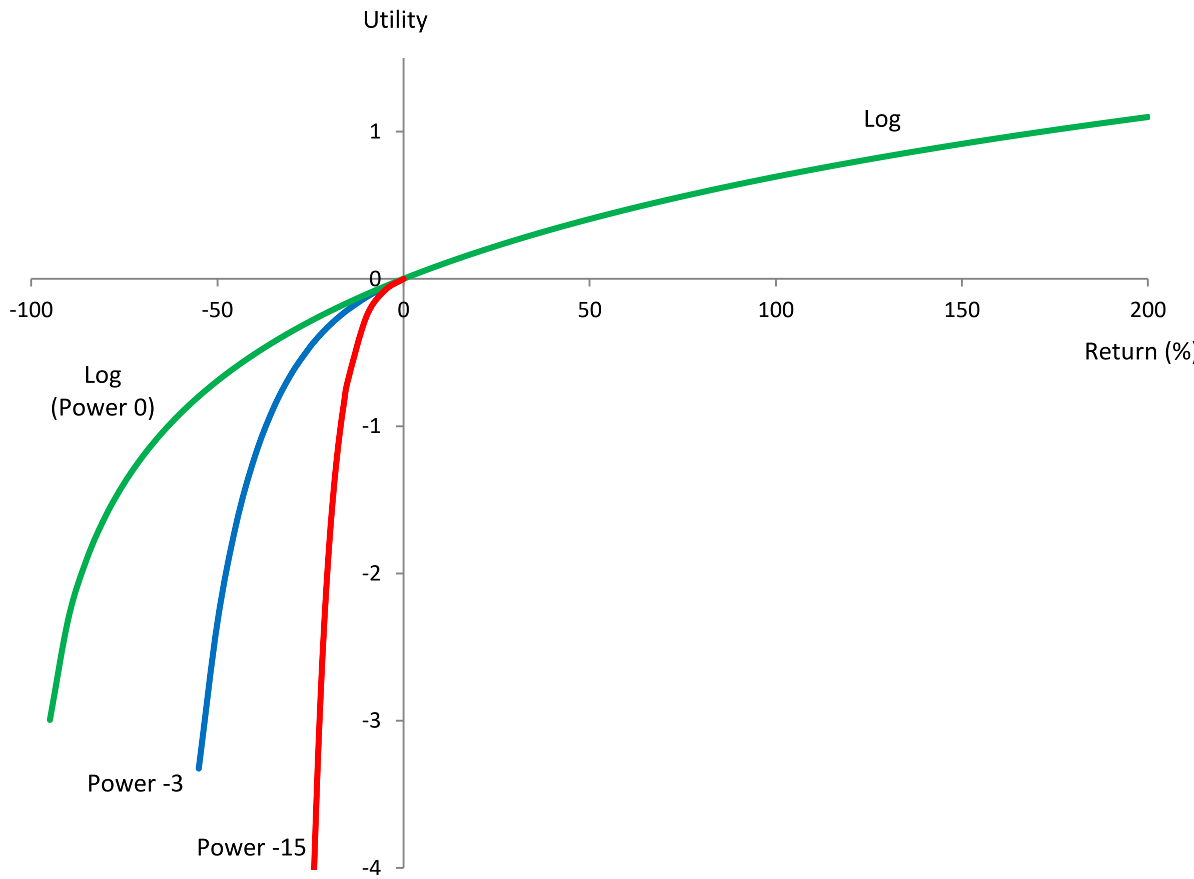

Fig. 3

S&P500 Monthly Return, 1950-2021

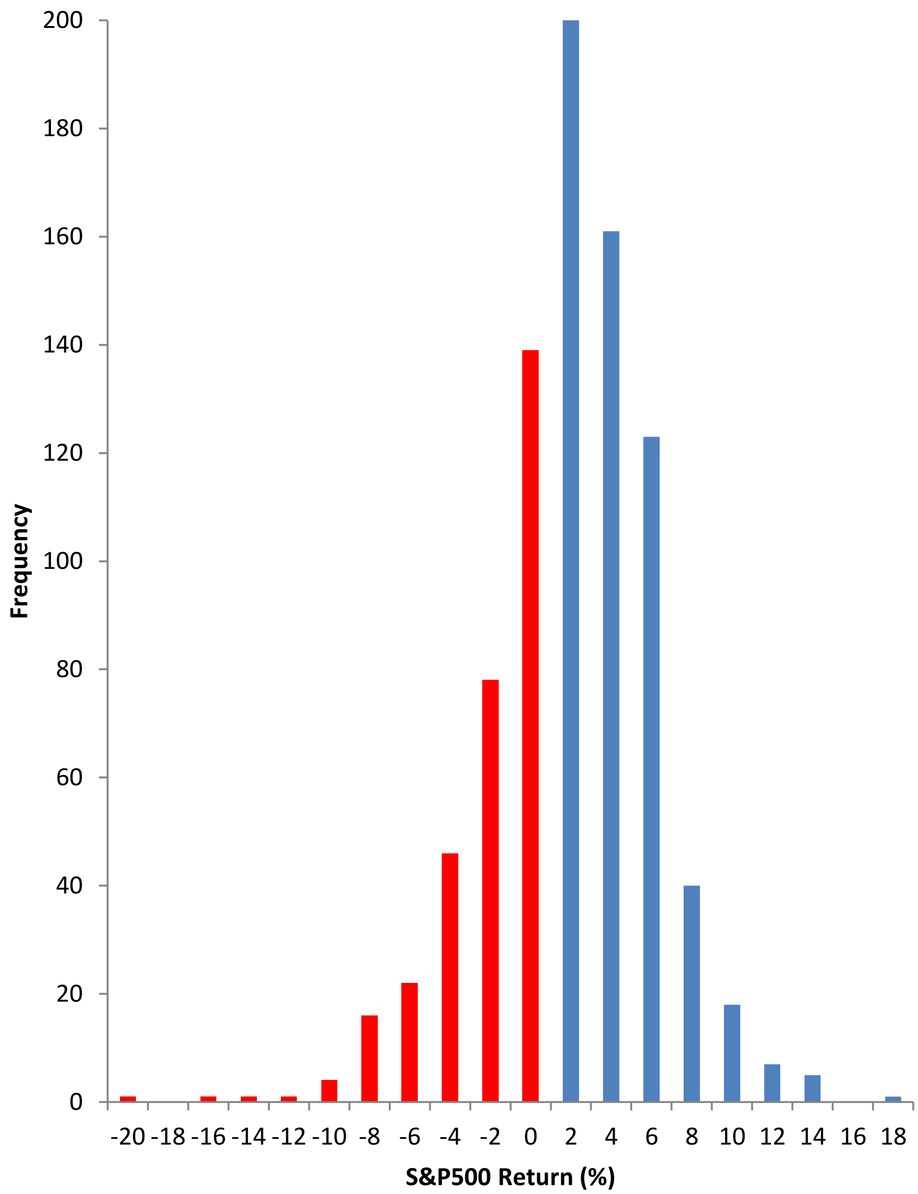

Fig. 4

Power-Log Utility OPM Prices for the ATM Call Option ($)

2:55 pm Eastern Standard Time, November 16, 2022

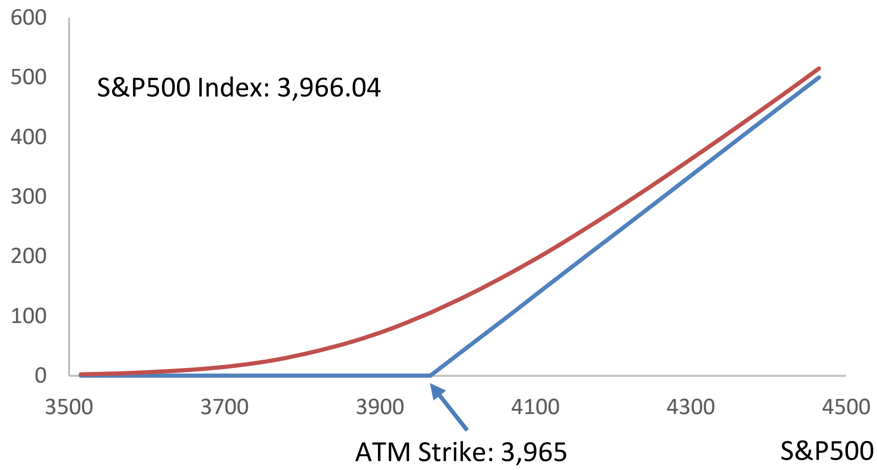

Fig. 5

S&P500 Call Option Price Differences from Market ($)

2:55 pm Eastern Standard Time, November 16, 2022

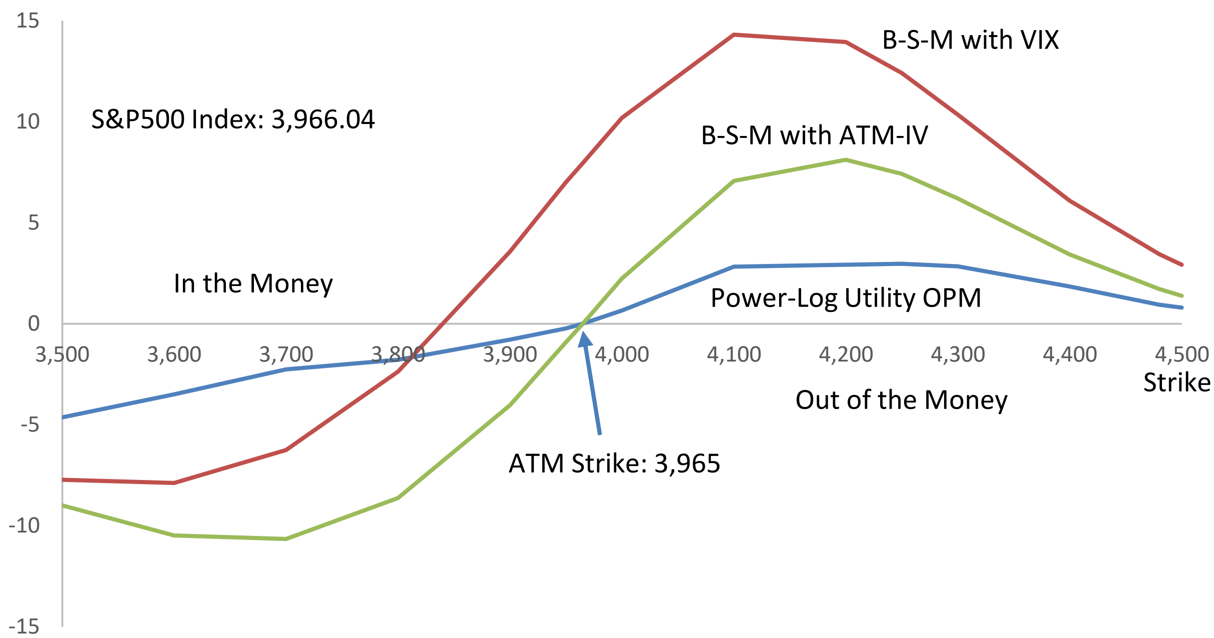

Fig. 6

Power-Log Utility OPM Prices for the ATM Put Option ($)

2:55 pm Eastern Standard Time, November 16, 2022

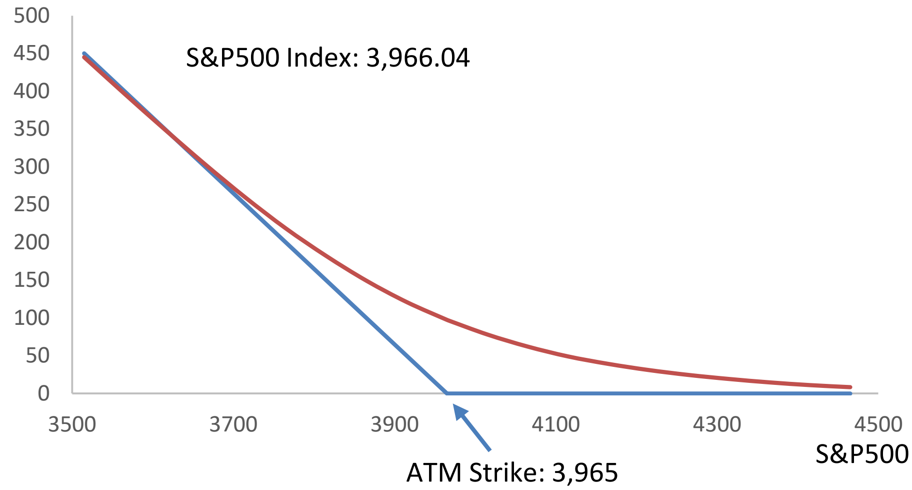

Fig. 7

S&P500 Put Option Price Differences from Market ($)

2:55 pm Eastern Standard Time, November 16, 2022